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Inverse Radiation Problem for Simultaneous Estimation of Temperature Profile and Surface Reflectivity

H. Y. Li* and M. N. Ozisik†
North Carolina State University, Raleigh, North Carolina 27695

A method is presented for simultaneous estimation of the unknown temperature distribution and the diffuse surface reflectivity in an absorbing, emitting, and isotropically scattering medium from the knowledge of the exit radiation intensities. The inverse radiation problem is recast as an optimization problem in finite-dimensional space and the conjugate gradient method of minimization is then used for its solution. The scheme is stable, insensitive to the initial guess, and in the absence of measurement errors the estimated solution converges to the exact result.

Nomenclature

a_n = expansion coefficients for the source term
$\mathbf{b}$ = $[a_0, a_1, \ldots, a_{N^*}, \rho]^T$
c_j = constants defined by Eq. (5b)
$\mathbf{d}$ = direction of descent
g_n = Chandrasekhar polynomials
I = radiation intensity
$\nabla \mathbf{I}$ = sensitivity coefficient vector
J = objective function
$\nabla \mathbf{J}$ = gradient of the objective function
$\bar{n}$ = refractive index
P_n = Legendre polynomials
S = source term defined by Eq. (1b)
T = temperature
Y = measured exit radiation intensities at the surface $\tau = 0$
Z = measured exit radiation intensities at the surface $\tau = \tau_0$
β = step size
γ = conjugate coefficient
ζ = random variables
μ = direction cosine
ξ = eigenvalues
ρ = surface reflectivity
σ = standard deviation
$\bar{\sigma}$ = Stefan-Boltzmann constant
τ = optical variable
τ_0 = optical thickness
ω = single-scattering albedo

Superscripts
k = kth iteration
T = transpose

Introduction

IN direct radiation problems, the equation of radiative transfer, the source term, and the properties of the medium and the boundary conditions are given; the radiation intensity is to be found.[1–3] In inverse radiation problems, the radiative properties, the source term, or the boundary conditions are to be determined from the knowledge of the measured exit radiation data. A considerable amount of work has been reported for determining the single-scattering albedo, the phase function, or the optical thickness of a medium from various types of exit radiation measurements.[4–12] The estimation of temperature profile in the atmosphere from the outgoing radiance is also reported.[13,14]

The conjugate gradient method has been used to solve linear equations,[15] minimization of functions with or without constraints,[16,17] and inverse heat conduction problems.[18–20] This iterative scheme converges very fast and is insensitive to the initial guess. In this work the conjugate gradient method of minimization is used to solve the inverse radiation problem for simultaneously determining the temperature distribution and the surface reflectivity in an absorbing, emitting, and isotropically scattering medium from the simulated exit intensities.

Analysis

The analysis, based on the finite-dimensional conjugate gradient method of minimization, consists of the following basic steps: 1) the direct problem; 2) the sensitivity problem; and 3) the gradient equation. We first describe the procedure for each of these steps and then present an algorithm for the solution of the inverse problem.

Direct Problem

For an absorbing, emitting, isotropically scattering, gray, plane-parallel medium of optical thickness τ_0, and azimuthally symmetric radiation, the equation of radiative transfer can be written as[1]

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = S(\tau) + \frac{\omega}{2} \int_{-1}^{1} I(\tau, \mu')\, \mathrm{d}\mu'$$

$$0 < \tau < \tau_0, \quad -1 < \mu < 1 \tag{1a}$$

with

$$S(\tau) = (1 - \omega) \frac{\bar{n}^2 \bar{\sigma} T^4(\tau)}{\pi} \tag{1b}$$

where $I(\tau, \mu)$ is the radiation intensity, $T(\tau)$ is the temperature, $\bar{n}$ is the refractive index, $\bar{\sigma}$ is the Stefan-Boltzmann constant, ω is the single scattering albedo, τ is the optical variable, μ is the cosine of the angle between the direction of the radiation intensity and the positive τ axis. For the case considered here, we assume that the boundary surface at $\tau = 0$ is a diffuse reflector and has negligible emission; the boundary surface at $\tau = \tau_0$ is transparent; there is no exter-

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*Graduate Student, Mechanical and Aerospace Engineering Department, P.O. Box 7910.
†Professor, Mechanical and Aerospace Engineering Department, P.O. Box 7910.

nally incident radiation. The boundary conditions become

$$I(0, \mu) = 2\rho \int_0^1 I(0, -\mu')\mu' \, d\mu' \qquad \mu > 0 \tag{1c}$$

$$I(\tau_0, -\mu) = 0 \qquad \mu > 0 \tag{1d}$$

where ρ, $0 \leq \rho \leq 1$, is the reflectivity of the boundary surface at $\tau = 0$.

The source term involving the fourth power of the temperature is approximated by a polynomial in τ as

$$S(\tau) = \sum_{n=0}^{N^*} a_n \tau^n \tag{2}$$

The above problem defined by Eqs. (1) will be referred to as the direct problem when $S(\tau)$, ω, ρ, and τ_0 are all specified and the radiation intensity is to be determined. Suppose $S(\tau)$ and ρ are unknown and are to be estimated simultaneously from the knowledge of the measured exit radiation intensities, then the problem becomes an inverse problem. Since any error involved in the solution of the direct problem effects the accuracy of the estimation, an accurate solution of the direct problem is essential in the analysis of the inverse problem. We present below the method of solution of the direct problem before describing the inverse analysis.

Using the P_N analysis, the solution of the direct problem given by Eqs. (1) for the radiation intensity $I(\tau, \mu)$ is expressed in the form[21]

$$I(\tau, \mu) = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \sum_{j=1}^{J^*} \{A_j e^{-\tau/\xi_j} + (-1)^n B_j \exp[-(\tau_0 - \tau)/\xi_j]\} g_n(\xi_j) + I_p(\tau, \mu) \tag{3}$$

where $J^* = (N + 1)/2$, N is an odd integer and $I_p(\tau, \mu)$ denotes a particular solution of Eq. (1a) corresponding to the inhomogeneous source term $S(\tau)$. Here $g_n(\xi)$ are determined from the recurrence formula

$$(n + 1)g_{n+1}(\xi) = h_n \xi g_n(\xi) - n g_{n-1}(\xi) \tag{4a}$$

for $n = 0, 1, \ldots, N$, with $g_0(\xi) = 1$ and $h_n = 2n + 1 - \omega\delta_{0n}$, where δ_{0n} is the Kronecker delta. The eigenvalues ξ_j, $j = 1, 2, \ldots, J^*$, are the J^* positive solutions of the following eigenvalue problem:

$$\frac{n(n-1)}{h_n h_{n-1}} g_{n-2}(\xi) + \frac{1}{h_n}\left[\frac{(n+1)^2}{h_{n+1}} + \frac{n^2}{h_{n-1}}\right] g_n(\xi) + \frac{(n+2)(n+1)}{h_{n+1}h_n} g_{n+2}(\xi) = \xi^2 g_n(\xi) \tag{4b}$$

for $n = 0, 2, 4, \ldots, N - 1$.

The particular solution of Eq. (1a) is[22]

$$I_p(\tau, \mu) = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \cdot \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} \left\{ \int_0^{\tau} S(\tau')\exp[-(\tau - \tau')/\xi_j] \, d\tau' + (-1)^n \int_{\tau}^{\tau_0} S(\tau')\exp[-(\tau' - \tau)/\xi_j] \, d\tau' \right\} g_n(\xi_j) \tag{5a}$$

where

$$c_j = \left[\sum_{n=1}^{J^*} g_{2n-2}^2(\xi_j) h_{2n-2}\right]^{-1} \tag{5b}$$

The constants A_j and B_j are determined by requiring the solution given by Eq. (3) to satisfy the Marshak boundary conditions as discussed in Ref. 1 as applied to the boundary conditions given by Eqs. (1c) and (1d).

Once A_j and B_j are available, the exit intensities at $\tau = 0$ and $\tau = \tau_0$ are calculated from[21]

$$\begin{aligned} I(0, -\mu) = {} & \sum_{n=0}^{N} (-1)^n \frac{2n+1}{2} P_n(-\mu) \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} g_n(\xi_j) \cdot \int_0^{\tau_0} S(\tau') e^{-\tau'/\xi_j} \, d\tau' - e^{-\tau_0/\mu} \sum_{n=0}^{N} \frac{2n+1}{2} \\ & \cdot P_n(-\mu) \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} g_n(\xi_j) \int_0^{\tau_0} S(\tau')\exp[-(\tau_0 - \tau')/\xi_j] \, d\tau' \\ & + \frac{\omega}{2} \sum_{j=1}^{J^*} \xi_j \left[A_j \frac{1 - e^{-\tau_0/\mu} e^{-\tau_0/\xi_j}}{\mu + \xi_j} + B_j \frac{e^{-\tau_0/\mu} - e^{-\tau_0/\xi_j}}{\mu - \xi_j} \right] \\ & \mu > 0 \end{aligned} \tag{6a}$$

$$\begin{aligned} I(\tau_0, \mu) = {} & \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} g_n(\xi_j) \cdot \int_0^{\tau_0} S(\tau')\exp[-(\tau_0 - \tau')/\xi_j] \, d\tau' \\ & - e^{-\tau_0/\mu} \sum_{n=0}^{N} \frac{2n+1}{2} [(-1)^n P_n(\mu) - 2\rho S_{0,n}] \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} g_n(\xi_j) \int_0^{\tau_0} S(\tau') e^{-\tau'/\xi_j} \, d\tau' \\ & + \rho\omega e^{-\tau_0/\mu} \sum_{j=1}^{J^*} \xi_j \left(A_j \int_0^1 \frac{1 - e^{-\tau_0/\mu'} e^{-\tau_0/\xi_j}}{\mu' + \xi_j} \mu' \, d\mu' + B_j \int_0^1 \frac{e^{-\tau_0/\mu'} - e^{-\tau_0/\xi_j}}{\mu' - \xi_j} \mu' \, d\mu' \right) - 2\rho e^{-\tau_0/\mu} \sum_{n=0}^{N} \frac{2n+1}{2} \\ & \cdot \left[\int_0^1 P_n(-\mu') e^{-\tau_0/\mu'} \mu' \, d\mu' \right] \sum_{j=1}^{J^*} \frac{c_j}{\xi_j} g_n(\xi_j) \cdot \int_0^{\tau_0} S(\tau')\exp[-(\tau_0 - \tau')/\xi_j] \, d\tau' + \frac{\omega}{2} \sum_{j=1}^{J^*} \xi_j \\ & \cdot \left[A_j \frac{e^{-\tau_0/\mu} - e^{-\tau_0/\xi_j}}{\mu - \xi_j} + B_j \frac{1 - e^{-\tau_0/\mu} e^{-\tau_0/\xi_j}}{\mu + \xi_j} \right] \qquad \mu > 0 \end{aligned} \tag{6b}$$

The inverse analysis is now concerned with simultaneous estimation of the coefficients $a_n (n = 0, 1, \ldots, N^*)$ of the source term defined by Eq. (2) and ρ of the boundary surface at $\tau = 0$ from the knowledge of the exit radiation intensities taken at both surfaces of the plate at several different polar directions. We denote $N^* + 2$ unknown parameters with a vector $\boldsymbol{b}$ defined as

$$\boldsymbol{b} = [a_0, a_1, \ldots, a_{N^*}, \rho]^T \tag{7}$$

In the solution of this inverse radiation problem with the conjugate gradient method of minimization, two quantities called the "sensitivity coefficients" and the "gradient equation" will be needed in the analysis. Therefore, before proceeding to the inverse analysis, we present below the development of the sensitivity problem for the determination of the sensitivity coefficients and the gradient equation.

Sensitivity Problem

The estimation of the unknown source term $S(\tau)$ and the boundary surface reflectivity ρ from the knowledge of the exit intensities measured at different directions can be recast as a problem of minimization of the following objective function $J(\boldsymbol{b})$:

$$J(\boldsymbol{b}) = \int_{-1}^{0} [I(0, \mu; \boldsymbol{b}) - Y(\mu)]^2 \, d\mu + \int_{0}^{1} [I(\tau_0, \mu; \boldsymbol{b}) - Z(\mu)]^2 \, d\mu \tag{8}$$

where $Y(\mu)$ and $Z(\mu)$ are the measured exit radiation intensities at the surfaces $\tau = 0$ and $\tau = \tau_0$, respectively, and $I(0, \mu; \boldsymbol{b})$ and $I(\tau_0, \mu; \boldsymbol{b})$ are the estimated exit radiation intensities at the surfaces $\tau = 0$ and $\tau = \tau_0$, respectively, using the estimated values of the coefficients $\boldsymbol{b} = [a_0, a_1, \ldots, a_{N^*}, \rho]^T$ where a_n are defined by Eq. (2). Therefore, the inverse radiation problem is reduced to an optimization problem in $(N^* + 2)$-dimensional space.

The gradient of the objective function is obtained by differentiating $J(\boldsymbol{b})$ given by Eq. (8) with respect to each of the unknown coefficients a_n and ρ, respectively. Then the resulting expressions for $\partial J/\partial a_n$ and $\partial J/\partial \rho$ contain the sensitivity coefficients $\partial I/\partial a_n$ and $\partial I/\partial \rho$, respectively, which can be determined from the solution of the sensitivity problem developed as described below.

Differentiating the direct problem given by Eqs. (1) with respect to a_n and ρ, respectively, then rearranging the result, the following sensitivity problem is obtained for the determination of the sensitivity coefficients $\partial I/\partial a_n$ and $\partial I/\partial \rho$, ($n = 0, 1, \ldots, N^*$):

$$\mu \frac{\partial}{\partial \tau}\left[\frac{\partial I(\tau, \mu)}{\partial a_n}\right] + \left[\frac{\partial I(\tau, \mu)}{\partial a_n}\right] = \tau^n + \frac{\omega}{2} \cdot \int_{-1}^{1} \left[\frac{\partial I(\tau, \mu')}{\partial a_n}\right] d\mu' \qquad 0 < \tau < \tau_0, \; -1 < \mu < 1 \tag{9a}$$

with boundary conditions

$$\left[\frac{\partial I(0, \mu)}{\partial a_n}\right] = 2\rho \int_0^1 \left[\frac{\partial I(0, -\mu')}{\partial a_n}\right] \mu' \, d\mu' \qquad \mu > 0 \tag{9b}$$

$$\left[\frac{\partial I(\tau_0, -\mu)}{\partial a_n}\right] = 0 \qquad \mu > 0 \tag{9c}$$

for $n = 0, 1, \ldots, N^*$

$$\mu \frac{\partial}{\partial \tau}\left[\frac{\partial I(\tau, \mu)}{\partial \rho}\right] + \left[\frac{\partial I(\tau, \mu)}{\partial \rho}\right] = \frac{\omega}{2} \int_{-1}^{1} \left[\frac{\partial I(\tau, \mu')}{\partial \rho}\right] d\mu' \qquad 0 < \tau < \tau_0, \; -1 < \mu < 1 \tag{10a}$$

with boundary conditions

$$\left[\frac{\partial I(0, \mu)}{\partial \rho}\right] = 2 \int_0^1 I(0, -\mu')\mu' \, d\mu' + 2\rho \int_0^1 \left[\frac{\partial I(0, -\mu')}{\partial \rho}\right] \mu' \, d\mu' \qquad \mu > 0 \tag{10b}$$

$$\left[\frac{\partial I(\tau_0, -\mu)}{\partial \rho}\right] = 0 \qquad \mu > 0 \tag{10c}$$

The row vector

$$\nabla \boldsymbol{I} = \left[\frac{\partial I}{\partial a_0}, \frac{\partial I}{\partial a_1}, \ldots, \frac{\partial I}{\partial a_{N^*}}, \frac{\partial I}{\partial \rho}\right] \tag{11}$$

is the sensitivity coefficient vector which can be determined from the solution of the sensitivity problem. We note that ρ being unknown, the products $\rho \partial I/\partial a_n$ and $\rho \partial I/\partial \rho$ appear in the boundary conditions Eq. (9b) and Eq. (10b), respectively, hence the estimation is nonlinear. If the reflectivity were known the estimation problem would be linear. Clearly, the solution procedure for Eqs. (9) is similar to that for the direct problem defined by Eqs. (1) with $S(\tau)$ replaced by τ^n, therefore, it will not be repeated here. We need to consider only the solution of Eqs. (10) for $\partial I/\partial \rho$ below. Using the P_N approximation, the solution is expressed in the form

$$\left[\frac{\partial I(\tau, \mu)}{\partial \rho}\right] = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) \cdot \sum_{j=1}^{J^*} \{A_j' e^{-\tau/\xi_j} + (-1)^n B_j' \exp[-(\tau_0 - \tau)/\xi_j]\} g_n(\xi_j) \tag{12}$$

The constants A_j' and B_j' are determined by requiring the solution given by Eq. (12) to satisfy the Marshak boundary conditions, i.e.

$$\sum_{n=0}^{N} \frac{2n+1}{2} \sum_{j=1}^{J^*} [S_{\alpha,n} - 2(-1)^n \rho S_{0,n} S_{\alpha,0}] \cdot [A_j' + (-1)^n B_j' e^{-\tau_0/\xi_j}] g_n(\xi_j) = 2S_{\alpha,0} \int_0^1 I(0, -\mu')\mu' \, d\mu' \tag{13a}$$

$$\sum_{n=0}^{N} \frac{2n+1}{2} S_{\alpha,n} \sum_{j=1}^{J^*} [(-1)^n A_j' e^{-\tau_0/\xi_j} + B_j'] g_n(\xi_j) = 0 \tag{13b}$$

for $\alpha = 0, 1, \ldots, (N-1)/2$, where

$$S_{\alpha,n} = \int_0^1 P_{2\alpha+1}(\mu) P_n(\mu) \, d\mu \tag{13c}$$

Once A_j' and B_j' are available, the required solutions for $\partial I/\partial \rho$ are computed from

$$\left[\frac{\partial I(0, -\mu)}{\partial \rho}\right] = \frac{\omega}{2} \sum_{j=1}^{J^*} \xi_j \left[A_j' \frac{1 - e^{-\tau_0/\mu} e^{-\tau_0/\xi_j}}{\mu + \xi_j} + B_j' \frac{e^{-\tau_0/\mu} - e^{-\tau_0/\xi_j}}{\mu - \xi_j}\right] \qquad \mu > 0 \tag{14a}$$

$$\begin{aligned}\left[\frac{\partial I(\tau_0, \mu)}{\partial \rho}\right] = {} & 2e^{-\tau_0/\mu} \int_0^1 I(0, -\mu')\mu' \, d\mu' \\ & + \rho \omega e^{-\tau_0/\mu} \sum_{j=1}^{J^*} \xi_j \left[A_j' \int_0^1 \frac{1 - e^{-\tau_0/\mu'} e^{-\tau_0/\xi_j}}{\mu' + \xi_j} \mu' \, d\mu' + B_j' \int_0^1 \frac{e^{-\tau_0/\mu'} - e^{-\tau_0/\xi_j}}{\mu' - \xi_j} \mu' \, d\mu'\right] \\ & + \frac{\omega}{2} \sum_{j=1}^{J^*} \xi_j \left[A_j' \frac{e^{-\tau_0/\mu} - e^{-\tau_0/\xi_j}}{\mu - \xi_j} + B_j' \frac{1 - e^{-\tau_0/\mu} e^{-\tau_0/\xi_j}}{\mu + \xi_j}\right] \qquad \mu > 0\end{aligned} \tag{14b}$$

Gradient Equation

Next, we develop expressions for the components of the gradient, i.e., $\partial J/\partial a_n$ and $\partial J/\partial \rho$, by differentiating J given by Eq. (8) with respect to a_n and ρ to obtain, respectively

$$\frac{\partial J}{\partial a_n} = \int_{-1}^{0} 2[I(0, \mu; \boldsymbol{b}) - Y(\mu)] \frac{\partial I(0, \mu; \boldsymbol{b})}{\partial a_n} d\mu + \int_0^1 2[I(\tau_0, \mu; \boldsymbol{b}) - Z(\mu)] \frac{\partial I(\tau_0, \mu; \boldsymbol{b})}{\partial a_n} d\mu \tag{15a}$$

for $n = 0, 1, \ldots, N^*$

$$\frac{\partial J}{\partial \rho} = \int_{-1}^{0} 2[I(0, \mu; \boldsymbol{b}) - Y(\mu)] \frac{\partial I(0, \mu; \boldsymbol{b})}{\partial \rho} \, d\mu + \int_{0}^{1} 2[I(\tau_0, \mu; \boldsymbol{b}) - Z(\mu)] \frac{\partial I(\tau_0, \mu; \boldsymbol{b})}{\partial \rho} \, d\mu \tag{15b}$$

where the row vector $\nabla \boldsymbol{J}$ defined by

$$\nabla \boldsymbol{J} = \left(\frac{\partial J}{\partial a_0}, \frac{\partial J}{\partial a_1}, \ldots, \frac{\partial J}{\partial a_{N^*}}, \frac{\partial J}{\partial \rho} \right) \tag{15c}$$

is the gradient of the objective function. The components $\partial J / \partial a_n$ and $\partial J / \partial \rho$ can be computed from Eqs. (15) since the sensitivity coefficients $\partial I / \partial a_n$ and $\partial I / \partial \rho$, the exit intensities, and the measured data $Y(\mu)$ and $Z(\mu)$ are available.

Conjugate Gradient Method of Minimization

To determine the unknown vector $\boldsymbol{b}$ defined by Eq. (7), we consider the following iterative process[18-20]:

$$\boldsymbol{b}^{k+1} = \boldsymbol{b}^k - \beta^k \boldsymbol{d}^k \tag{16}$$

where β^k is the "step size," $\boldsymbol{d}^k$ is the "direction of descent" at kth iteration which is determined from

$$\boldsymbol{d}^k = \nabla \boldsymbol{J}^T(\boldsymbol{b}^k) + \gamma^k \boldsymbol{d}^{k-1} \tag{17}$$

The "conjugate coefficient" γ^k is computed from

$$\gamma^k = \frac{\nabla \boldsymbol{J}(\boldsymbol{b}^k) \nabla \boldsymbol{J}^T(\boldsymbol{b}^k)}{\nabla \boldsymbol{J}(\boldsymbol{b}^{k-1}) \nabla \boldsymbol{J}^T(\boldsymbol{b}^{k-1})} \tag{18}$$

with $\gamma^0 = 0$. Here, the step size β^k in going from $\boldsymbol{b}^k$ to $\boldsymbol{b}^{k+1}$ is determined from the condition $\min_\beta J(\boldsymbol{b}^{k+1})$ or $\min_\beta J(\boldsymbol{b}^k - \beta^k \boldsymbol{d}^k)$, i.e.

$$\min_\beta J(\boldsymbol{b}^{k+1}) = \min_\beta \left\{ \int_{-1}^{0} [I(0, \mu; \boldsymbol{b}^k - \beta^k \boldsymbol{d}^k) - Y(\mu)]^2 \, d\mu + \int_{0}^{1} [I(\tau_0, \mu; \boldsymbol{b}^k - \beta^k \boldsymbol{d}^k) - Z(\mu)]^2 \, d\mu \right\} \tag{19a}$$

Using Taylor series expansion and keeping only the linear terms, Eq. (19a) becomes

$$\min_\beta J(\boldsymbol{b}^{k+1}) = \min_\beta \left\{ \int_{-1}^{0} [I(0, \mu; \boldsymbol{b}^k) - \beta^k \nabla \boldsymbol{I}(0, \mu; \boldsymbol{b}^k) \boldsymbol{d}^k - Y(\mu)]^2 \, d\mu + \int_{0}^{1} [I(\tau_0, \mu; \boldsymbol{b}^k) - \beta^k \nabla \boldsymbol{I}(\tau_0, \mu; \boldsymbol{b}^k) \boldsymbol{d}^k - Z(\mu)]^2 \, d\mu \right\} \tag{19b}$$

To minimize Eq. (19b) we differentiate it with respect to β^k and set the result equal to zero. Solving the resulting equation for β^k, the following expression is obtained:

$$\beta^k = \frac{\int_{-1}^{0} [I(0, \mu; \boldsymbol{b}^k) - Y(\mu)] \nabla \boldsymbol{I}(0, \mu; \boldsymbol{b}^k) \, \boldsymbol{d}^k \, d\mu + \int_{0}^{1} [I(\tau_0, \mu; \boldsymbol{b}^k) - Z(\mu)] \nabla \boldsymbol{I}(\tau_0, \mu; \boldsymbol{b}^k) \, \boldsymbol{d}^k \, d\mu}{\int_{-1}^{0} [\nabla \boldsymbol{I}(0, \mu; \boldsymbol{b}^k) \boldsymbol{d}^k]^2 \, d\mu + \int_{0}^{1} [\nabla \boldsymbol{I}(\tau_0, \mu; \boldsymbol{b}^k) \boldsymbol{d}^k]^2 \, d\mu} \tag{20}$$

Once $\boldsymbol{d}^k$ is calculated from Eq. (17) and β^k from Eq. (20), the iterative process defined by Eq. (16) can be used to determine $\boldsymbol{b}^{k+1}$ to satisfy a specified stopping criterion. If the problem involves no measurement errors, the condition

$$J(\boldsymbol{b}^{k+1}) < \delta^* \tag{21}$$

could be used for terminating the iterative process, where δ^* is a small positive number. However, the measured radiation intensities contain measurement errors. Computational experiments suggest the use of the discrepancy principle

$$J(\boldsymbol{b}^{k+1}) < 2\sigma^2 \tag{22}$$

as the stopping criterion,[18,23] where σ is the standard deviation of the measurement errors.

Computational Algorithm

The iterative procedure for the conjugate gradient method can be summarized as follows: Assume $\boldsymbol{b}^k$ is known at the kth iteration.

1) Solve the direct problem Eqs. (1), and compute the exit radiation intensities $I(0, \mu; \boldsymbol{b}^k)$ and $I(\tau_0, \mu; \boldsymbol{b}^k)$ at the surfaces $\tau = 0$ and $\tau = \tau_0$, respectively.

2) Solve the sensitivity problem Eqs. (9) and (10), and compute the sensitivity coefficient vector $\nabla \boldsymbol{I}$ defined by Eq. (11).

3) Knowing $\nabla \boldsymbol{I}$, $I(0, \mu; \boldsymbol{b}^k)$, $I(\tau_0, \mu; \boldsymbol{b}^k)$, and the measured exit radiation intensities $Y(\mu)$ and $Z(\mu)$, compute the gradient $\nabla \boldsymbol{J}(\boldsymbol{b}^k)$ from Eqs. (15).

4) Knowing $\nabla \boldsymbol{J}(\boldsymbol{b}^k)$, compute the conjugate coefficient γ^k from Eq. (18), then compute the direction of descent $\boldsymbol{d}^k$ from Eq. (17).

5) Knowing $\nabla \boldsymbol{I}(0, \mu; \boldsymbol{b}^k)$, $I(\tau_0, \mu; \boldsymbol{b}^k)$, $Y(\mu)$, $Z(\mu)$, and $\boldsymbol{d}^k$, compute the step size β^k from Eq. (20).

6) Knowing β^k and $\boldsymbol{d}^k$, compute $\boldsymbol{b}^{k+1}$ from Eq. (16).

7) Terminate the process if a specified stopping criterion is satisfied, otherwise return to step 1.

To initiate the iteration an initial guess $\boldsymbol{b}^0 = \boldsymbol{0}$ is used.

Results and Discussions

We now present numerical results to demonstrate the accuracy of the conjugate gradient method of minimization for simultaneously estimating the spatial distribution of the unknown source term $S(\tau)$ defined by Eq. (2) which is related to $T(\tau)$ and ρ from the knowledge of the exit radiation intensities. For combustion problems encountered in fires and furnaces, the temperatures are in general between 800–1800 K.[24] Therefore, the source terms are selected such that the temperatures lie in this range. In coal flames, the single-scattering albedo is in the range 0.2–0.35.[25] With these considerations we have chosen the values of the single-scattering albedo and the refractive index to be 0.3 and 1, respectively. In order to simulate Y and Z containing measurement errors, random errors of σ are added to the exact exit intensities computed from the solution of the direct problem. Thus, we have

$$Y_{\text{measured}} = Y_{\text{exact}} + \sigma\zeta \tag{23a}$$

$$Z_{\text{measured}} = Z_{\text{exact}} + \sigma\zeta \tag{23b}$$

For normally distributed errors there is a 99% probability of a value of ζ lying in the range $-2.576 < \zeta < 2.576$. A random number generator such as the IMSL subroutine DRNNOR can be used to generate values of ζ. For all the results presented in this work the exit radiation intensities are measured at the surfaces $\tau = 0$ and $\tau = \tau_0$, and 20 measurement points are taken at each surface over the polar angle interval $0 \leq \theta \leq \pi/2$.

First, we examine the accuracy of the estimation of the source term and surface reflectivity from the knowledge of the exit radiation intensities containing no measurement errors, i.e., $\sigma = 0$. The estimated solutions obtained with the inverse analysis converged to the exact values for all the cases considered here.

Next, we consider the inverse analysis using the exit intensities containing measurement errors. Figure 1 is for a source term expressed in a polynomial of degree four in the optical variable τ as

$$S(\tau) = 1 + 10\tau + 75\tau^2 - 170\tau^3 + 85\tau^4 \text{ W/cm}^2 \quad \text{in} \quad 0 \leq \tau \leq 1 \tag{24}$$

and the surface reflectivity taken as $\rho = 0.9$. The exit intensities with measurement errors $\sigma = 0.05$ are used to estimate $S(\tau)$ given by Eq. (24) and the ρ with the inverse analysis for the case of $\omega = 0.3$, $\tau_0 = 1$ and $\bar{n} = 1$. The resulting estimated source term and the temperature distribution are shown in Fig. 1. Clearly, the agreement between the exact and the estimated results is very good.

Figure 2 is prepared to illustrate the effects of the standard deviation of measurement errors on the accuracy of the estimation. Therefore, Fig. 2 is similar to Fig. 1 except the standard deviation for the measurement errors in the exit intensities is taken as $\sigma = 0.1$. Increasing σ from 0.05 to 0.1, as expected, decreases the accuracy of the estimation, but the estimated results are still good.

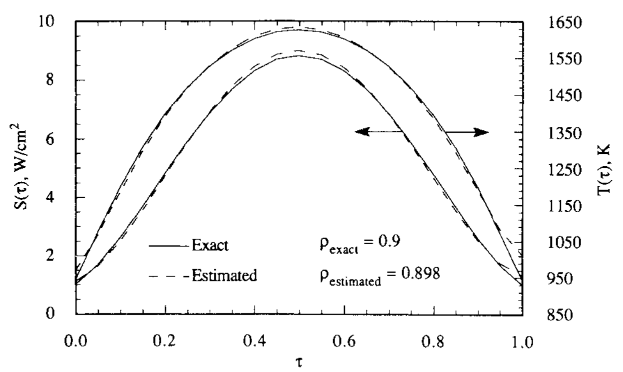

Fig. 1 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.05$, $S(\tau) = 1 + 10\tau + 75\tau^2 - 170\tau^3 + 85\tau^4$ W/cm², $\omega = 0.3$, $\tau_0 = 1$, $\bar{n} = 1$.

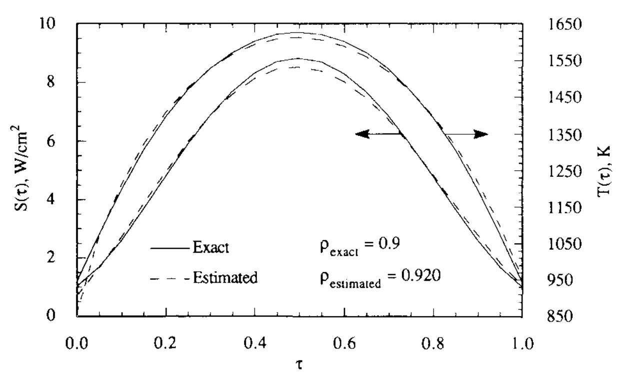

Fig. 2 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.1$, $S(\tau) = 1 + 10\tau + 75\tau^2 - 170\tau^3 + 85\tau^4$ W/cm², $\omega = 0.3$, $\tau_0 = 1$, $\bar{n} = 1$.

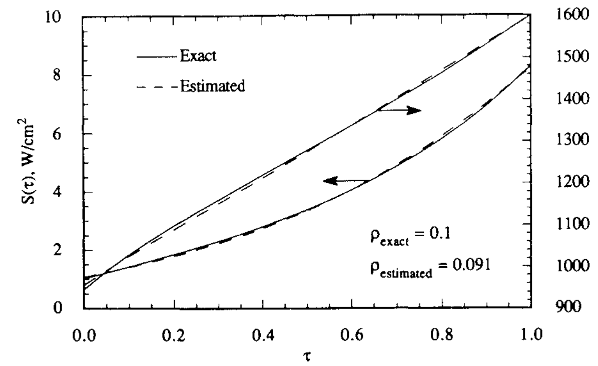

Fig. 3 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.05$, $S(\tau) = 1 + 2$ Sinh(2τ) W/cm², $\omega = 0.3$, $\tau_0 = 1$, $\bar{n} = 1$.

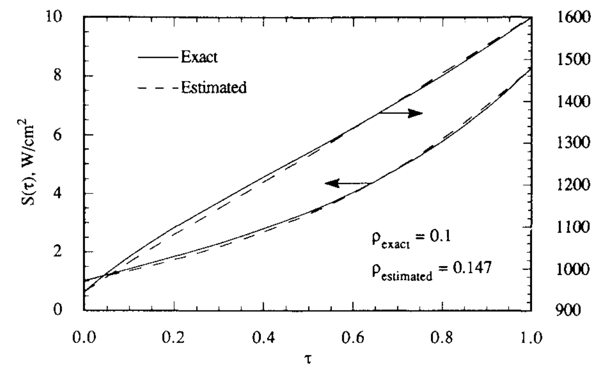

Fig. 4 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.1$, $S(\tau) = 1 + 2$ Sinh(2τ) W/cm², $\omega = 0.3$, $\tau_0 = 1$, $\bar{n} = 1$.

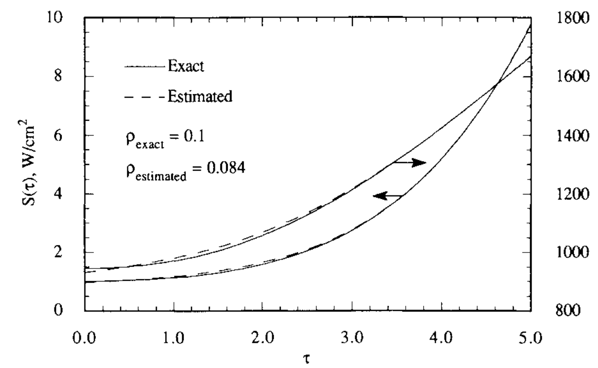

Fig. 5 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.05$, $S(\tau) = 1 + 0.1\tau^2 + 0.01\tau^4$ W/cm², $\omega = 0.3$, $\tau_0 = 5$, $\bar{n} = 1$.

Figures 3 and 4 are prepared to illustrate the effects of the functional form of the source term. A source term in the form of a hyperbolic sine given by

$$S(\tau) = 1 + 2\,\text{Sinh}(2\tau) \text{ W/cm}^2 \quad \text{in} \quad 0 \leq \tau \leq 1 \tag{25}$$

is chosen and the surface reflectivity is taken as $\rho = 0.1$, while $\omega = 0.3$, $\tau_0 = 1$ and $\bar{n} = 1$ remained the same as before. Figures 3 and 4 show the results obtained with the inverse analysis for simulated experimental data containing measurement errors of $\sigma = 0.05$ and $\sigma = 0.1$, respectively. The estimated results remain still good.

Figures 5 and 6 are presented to show the effects of the optical thickness to the estimation. In this case, the unknown source term is represented as a polynomial of degree four in the optical variable over the optical thickness $\tau_0 = 5$, i.e.

$$S(\tau) = 1 + 0.1\tau^2 + 0.01\tau^4 \text{ W/cm}^2 \quad \text{in} \quad 0 \leq \tau \leq 5 \tag{26}$$

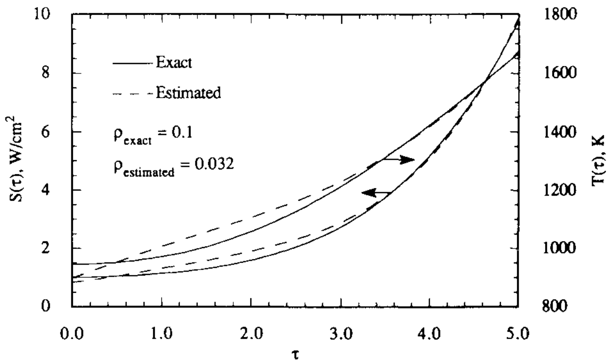

Fig. 6 Simultaneous estimation of the spatial variation of the source term and the surface reflectivity with measurement errors $\sigma = 0.1$, $S(\tau) = 1 + 0.1\tau^2 + 0.01\tau^4$ W/cm², $\omega = 0.3$, $\tau_0 = 5$, $\bar{n} = 1$.

and the unknown ρ is taken as 0.1. Figures 5 and 6 show the estimated source term and temperature distribution for $\sigma = 0.05$ and $\sigma = 0.1$, respectively. The estimation becomes more sensitive to the increased measurement errors (i.e., $\sigma = 0.1$) for the case with a larger optical thickness (i.e., $\tau_0 = 5$).

Conclusions

The conjugate gradient method is used to solve the inverse radiation problem for simultaneous estimation of the temperature distribution and surface reflectivity by using the simulated exit radiation intensities. Even though the temperature profiles are functions of τ, only exit radiation intensities are needed as input data (i.e., no measurements are required inside the medium) to estimate the spatial variation of $T(\tau)$. With no measurement errors the estimated solutions converge to the exact values. The estimation becomes more sensitive to the measurement errors as the optical thickness is increased.

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